

# On Networks with Active and Passive Agents

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**Abstract**—We introduce an active-passive networked multi-agent system framework, which consists of agents subject to exogenous inputs (active agents) and agents without any inputs (passive agents), and analyze its convergence using Lyapunov stability.

## A. Preliminaries

In the multiagent literature, graphs are broadly adopted to encode interactions in networked systems [1], [2]. An *undirected* graph  $\mathcal{G}$  is defined by a set  $\mathcal{V}_{\mathcal{G}} = \{1, \dots, n\}$  of *nodes* and a set  $\mathcal{E}_{\mathcal{G}} \subset \mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{G}}$  of *edges*. If  $(i, j) \in \mathcal{E}_{\mathcal{G}}$ , then the nodes  $i$  and  $j$  are *neighbors* and the neighboring relation is indicated with  $i \sim j$ . The *degree* of a node is given by the number of its neighbors. Letting  $d_i$  be the degree of node  $i$ , then the *degree* matrix of a graph  $\mathcal{G}$ ,  $\mathcal{D}(\mathcal{G}) \in \mathbb{R}^{n \times n}$ , is given by  $\mathcal{D}(\mathcal{G}) \triangleq \text{diag}(d)$ ,  $d = [d_1, \dots, d_n]^T$ . A *path*  $i_0 i_1 \dots i_L$  is a finite sequence of nodes such that  $i_{k-1} \sim i_k$ ,  $k = 1, \dots, L$ , and a graph  $\mathcal{G}$  is *connected* if there is a path between any pair of distinct nodes. The *adjacency* matrix of a graph  $\mathcal{G}$ ,  $\mathcal{A}(\mathcal{G}) \in \mathbb{R}^{n \times n}$ , is given by

$$[\mathcal{A}(\mathcal{G})]_{ij} \triangleq \begin{cases} 1, & \text{if } (i, j) \in \mathcal{E}_{\mathcal{G}}, \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

The *Laplacian* matrix of a graph,  $\mathcal{L}(\mathcal{G}) \in \overline{\mathbb{S}}_+^{n \times n}$ , playing a central role in many graph theoretic treatments of multiagent systems, is given by

$$\mathcal{L}(\mathcal{G}) \triangleq \mathcal{D}(\mathcal{G}) - \mathcal{A}(\mathcal{G}). \quad (2)$$

Throughout this note, we model a given multiagent system by a connected, undirected graph  $\mathcal{G}$ , where nodes and edges represent agents and inter-agent communication links, respectively.

## B. Problem Formulation

Consider a system of  $n$  agents exchanging information among each other using their local measurements according to a connected, undirected graph  $\mathcal{G}$ . In addition, consider that

there exists  $m \geq 1$  exogenous inputs that interact with this system. We make the following definitions.

**Definition 1.** If agent  $i$ ,  $i = 1, \dots, n$ , is subject to one or more exogenous inputs (resp., no exogenous inputs), then it is an active agent (resp., passive agent).

**Definition 2.** If an exogenous input interacts with only one agent (resp., multiple agents), then it is an isolated input (resp., non-isolated input).

In this note, we are interested in the problem of driving the states of all (active and passive) agents to the average of the applied exogenous inputs. Motivating from this standpoint, we propose an integral action-based distributed control approach given by

$$\begin{aligned} \dot{x}_i(t) &= - \sum_{i \sim j} (x_i(t) - x_j(t)) + \sum_{i \sim j} (\xi_i(t) - \xi_j(t)) \\ &\quad - \sum_{i \sim h} (x_i(t) - c_h), \quad x_i(0) = x_{i0}, \end{aligned} \quad (3)$$

$$\dot{\xi}_i(t) = - \sum_{i \sim j} (x_i(t) - x_j(t)), \quad \xi_i(0) = \xi_{i0}, \quad (4)$$

where  $x_i(t) \in \mathbb{R}$  and  $\xi_i(t) \in \mathbb{R}$  denote the state and the integral action of agent  $i$ ,  $i = 1, \dots, n$ , respectively, and  $c_h \in \mathbb{R}$ ,  $h = 1, \dots, m$ , denotes an exogenous input applied to this agent. Similar to the  $i \sim j$  notation indicating the neighboring relation between agents, we use  $i \sim h$  to indicate the exogenous inputs that an agent is subject to.

Next, let  $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T \in \mathbb{R}^n$ ,  $\xi(t) = [\xi_1(t), \xi_2(t), \dots, \xi_n(t)]^T \in \mathbb{R}^n$ , and  $c = [c_1, c_2, \dots, c_m, 0, \dots, 0] \in \mathbb{R}^n$ , where we assume  $m \leq n$  for the ease of the following notation and without loss of generality. We can now write (3) and (4) in a compact form as

$$\begin{aligned} \dot{x}(t) &= -\mathcal{L}(\mathcal{G})x(t) + \mathcal{L}(\mathcal{G})\xi(t) - K_1 x(t) + K_2 c, \\ x(0) &= x_0, \end{aligned} \quad (5)$$

$$\dot{\xi}(t) = -\mathcal{L}(\mathcal{G})x(t), \quad \xi(0) = \xi_0, \quad (6)$$

where  $\mathcal{L}(\mathcal{G}) \in \overline{\mathbb{S}}_+^{n \times n}$ ,

$$K_1 \triangleq \text{diag}([k_{1,1}, k_{1,2}, \dots, k_{1,n}]^T) \in \overline{\mathbb{S}}_+^{n \times n}, \quad (7)$$

with  $k_{1,i} \in \overline{\mathbb{Z}}_+$  denoting the number of the exogenous inputs

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applied to agent  $i$ ,  $i = 1, \dots, n$ , and

$$K_2 \triangleq \begin{bmatrix} k_{2,11} & k_{2,12} & \cdots & k_{2,1n} \\ k_{2,21} & k_{2,22} & \cdots & k_{2,2n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{2,n1} & k_{2,n2} & \cdots & k_{2,nn} \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad (8)$$

with  $k_{2,ih} = 1$  if the exogenous input  $c_h(t)$ ,  $h = 1, \dots, m$ , is applied to agent  $i$ ,  $i = 1, \dots, n$ , and  $k_{2,ih} = 0$  otherwise. Note that  $k_{1,i} = \sum_{j=1}^n k_{2,ij}$ .

Since we are interested in driving the states of all (active and passive) agents to the average of the applied exogenous inputs, let

$$\delta(t) \triangleq x(t) - \epsilon \mathbf{1}_n \in \mathbb{R}^n, \quad (9)$$

$$\epsilon \triangleq \frac{\mathbf{1}_n^T K_2 c}{\mathbf{1}_n^T K_2 \mathbf{1}_n} \in \mathbb{R}, \quad (10)$$

be the error between  $x_i(t)$ ,  $i = 1, \dots, n$ , and the average of the applied exogenous inputs  $\epsilon$ . Based on (10),  $\epsilon$  can be equivalently written as

$$\begin{aligned} \epsilon &= \left( k_{2,11}c_1 + k_{2,12}c_2 + \cdots + k_{2,21}c_1 \right. \\ &\quad \left. + k_{2,22}c_2 + \cdots \right) / \left( k_{2,11} + k_{2,12} \right. \\ &\quad \left. + \cdots + k_{2,21} + k_{2,22} + \cdots \right), \end{aligned} \quad (11)$$

which is the average of the applied exogenous inputs.

### C. Convergence Analysis

It follows from (9) and  $\mathcal{L}(\mathcal{G})\mathbf{1}_n = \mathbf{0}_n$  of Lemma 1 that

$$\begin{aligned} \dot{\delta}(t) &= -\mathcal{L}(\mathcal{G})[\delta(t) + \epsilon \mathbf{1}_n] + \mathcal{L}(\mathcal{G})\xi(t) - K_1[\delta(t) \\ &\quad + \epsilon \mathbf{1}_n] + K_2 c(t) \\ &= -\mathcal{F}(\mathcal{G})\delta(t) + \mathcal{L}(\mathcal{G})\xi(t) - [K_1 \mathbf{1}_n \epsilon - K_2 c] \\ &= -\mathcal{F}(\mathcal{G})\delta(t) + \mathcal{L}(\mathcal{G})\xi(t) - \left[ \frac{K_1 \mathbf{1}_n \mathbf{1}_n^T K_2 c}{\mathbf{1}_n^T K_2 \mathbf{1}_n} - K_2 c \right] \\ &= -\mathcal{F}(\mathcal{G})\delta(t) + \mathcal{L}(\mathcal{G})\xi(t) - L_c K_2 c, \end{aligned} \quad (12)$$

where  $\mathcal{F}(\mathcal{G}) \triangleq \mathcal{L}(\mathcal{G}) + K_1$  and

$$L_c \triangleq \frac{K_1 \mathbf{1}_n \mathbf{1}_n^T}{\mathbf{1}_n^T K_2 \mathbf{1}_n} - \mathbf{I}_n. \quad (13)$$

Note that  $\mathcal{F}(\mathcal{G}) \in \mathbb{S}_+^{n \times n}$  and

$$\mathbf{1}_n^T L_c = \mathbf{1}_n^T \left[ \frac{K_1 \mathbf{1}_n \mathbf{1}_n^T}{\mathbf{1}_n^T K_2 \mathbf{1}_n} - \mathbf{I}_n \right] = \frac{\mathbf{1}_n^T K_1 \mathbf{1}_n}{\mathbf{1}_n^T K_2 \mathbf{1}_n} \mathbf{1}_n^T - \mathbf{1}_n^T = 0, \quad (14)$$

since  $(\mathbf{1}_n^T K_1 \mathbf{1}_n) / (\mathbf{1}_n^T K_2 \mathbf{1}_n) = 1$  from  $k_{1,i} = \sum_{j=1}^n k_{2,ij}$ .

Next, letting

$$e(t) \triangleq \xi(t) - \mathcal{L}^\dagger(\mathcal{G})L_c K_2 c, \quad (15)$$

and using (15) in (12) yields

$$\begin{aligned} \dot{\delta}(t) &= -\mathcal{F}(\mathcal{G})\delta(t) + \mathcal{L}(\mathcal{G})[e(t) + \mathcal{L}^\dagger(\mathcal{G})L_c K_2 c] - L_c K_2 c \\ &= -\mathcal{F}(\mathcal{G})\delta(t) + \mathcal{L}(\mathcal{G})e(t) + \left[ \mathbf{I}_n - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right] L_c K_2 c \\ &\quad - L_c K_2 c \\ &= -\mathcal{F}(\mathcal{G})\delta(t) + \mathcal{L}(\mathcal{G})e(t), \end{aligned} \quad (16)$$

since  $\frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T L_c K_2 c = 0$  as a direct consequence of (14). In addition, differentiating (15) with respect to time yields

$$\begin{aligned} \dot{e}(t) &= -\mathcal{L}(\mathcal{G})[\delta(t) + \epsilon \mathbf{1}_n] \\ &= -\mathcal{L}(\mathcal{G})\delta(t), \end{aligned} \quad (17)$$

where  $\mathcal{L}(\mathcal{G})\mathbf{1}_n = \mathbf{0}_n$ . The following theorem shows that the state of all agents  $x_i(t)$ ,  $i = 1, \dots, n$  asymptotically converge to  $\epsilon$ .

**Theorem 1.** Consider the networked multiagent system given by (3) and (4), where agents exchange information using local measurements and with  $\mathcal{G}$  defining a connected, undirected graph topology. Then, the closed-loop error dynamics defined by (16) and (17) are Lyapunov stable for all initial conditions and  $\delta(t)$  asymptotically vanishes.

*Proof.* Proof follows by considering Lyapunov function candidate given by  $V(\delta, e) = \frac{1}{2} \delta^T \delta + \frac{1}{2} e^T e$  and differentiating it along the trajectories of (16) and (17).  $\square$

Note that a generalized version of the proposed integral action-based distributed control approach can be given by

$$\begin{aligned} \dot{x}_i(t) &= -\alpha \sum_{i \sim j} (x_i(t) - x_j(t)) + \sum_{i \sim j} (\xi_i(t) - \xi_j(t)) \\ &\quad - \alpha \sum_{i \sim h} (x_i(t) - c_h), \quad x_i(0) = x_{i0}, \end{aligned} \quad (18)$$

$$\dot{\xi}_i(t) = -\gamma \sum_{i \sim j} (x_i(t) - x_j(t)), \quad \xi_i(0) = \xi_{i0}, \quad (19)$$

where  $\alpha \in \mathbb{R}_+$  and  $\gamma \in \mathbb{R}_+$ .

### D. Concluding Remarks

We investigated a system consisting of agents subject to exogenous constant inputs and agents without any inputs. Future research will consider extensions to time-varying exogenous inputs and more general graph topologies.

### REFERENCES

- [1] M. Mesbahi and M. Egerstedt, "Graph Theoretic Methods in Multi-agent Networks," *Princeton University Press*, 2010.
- [2] R. Olfati-Saber, J. Fax, and R. M. Murray, "Consensus and Cooperation in Networked Multiagent Systems," *Proc. of the IEEE*, vol. 95, pp. 215–233, 2007.